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ESSAY

An Algorithm for Extracting Square Roots

A traditional digit-by-digit square-root algorithm, explained through worked examples and justified by the expansion of a squared sum.

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There are various methods for extracting the square root of a number with any number of digits, including by hand.

We offer a well-known but undeservedly forgotten method that was once taught at school.

A calculator makes finding a square root easy: enter the number and press the square-root key. Calculators, however, are usually prohibited in tests and examinations. Tables of squares are sometimes allowed, but they are no help if the number is not a perfect square or has many digits. Finding a root by factorization is also laborious.

The algorithm through examples

The algorithm is based on the formula for the square of a sum:

$$(a + b)^2 = a^2 + 2ab + b^2$$

It finds the root successively, digit by digit. If the number is a perfect square, the process terminates. If it is not, the algorithm finds the root to any required accuracy.

The working is set out much like long division. It is sometimes called a “column calculation”.

Let us begin. We will first explain it through examples, then try to justify it theoretically.

Everyone knows the squares of single-digit numbers, which themselves have one or two digits. Thus the roots of one- or two-digit perfect squares are easily calculated mentally.

If the number has three or four digits, its square root has two digits. We will start there.

Example 1. Find the square root of 2916

$$\begin{aligned}\sqrt{2916} &= 54, \\ 29 - 5^2 &= 29 - 25 = 4, \\ 100 \cdot 4 + 16 &= 416, \\ 104 \cdot 4 &= 416, \\ 416 - 416 &= 0.\end{aligned}$$

1. Group the digits in pairs, marking them from right to left: 29'16.

The resulting groups are called periods, and the operation is called separating the number into periods. Their number tells us how many digits to expect in the root. Here, the root will have two digits because the original number has two periods.

2. Find the square root of 29 to integer accuracy, obtaining 5. Write 5 after the equals sign. This is the first digit of the root.
3. Square 5 and write the result below 29, the first period. Subtract 25 from 29, obtaining 4, and write it below 25.
4. Bring down the remaining digits, the second period, 16. This gives 416.
5. Now find the next digit of the root. Double the first digit, 5, obtaining 10.

To make finding the second digit easier, draw a vertical line to the left of the remainder and write the following operations beyond it. Append another digit to 10; this will become the next digit of the root. Multiply the resulting number by that same appended digit. The product must equal the remainder 416, or be as close as possible without exceeding it.

Try the digit 4. Append it to 10 and multiply 104 by 4. The result is 416, exactly our remainder. Thus the appended digit 4 is the next digit of the root. Return to the original calculation and write 4 after 5 in the answer.

6. Write 416, the product of 104 and 4, below the remainder 416.

Perform the subtraction $416 - 416 = 0$. A zero remainder tells us the calculation is complete.

Answer: $\sqrt{2916} = 54$

Can you justify the steps algebraically or geometrically?

Let us continue. If the number has five or six digits, its square root has three digits.

Example 2. Calculate the square root of 56169

$$\begin{aligned}\sqrt{56169} &= 237, \\ 5 - 2^2 &= 5 - 4 = 1, \\ 100 \cdot 1 + 61 &= 161, \\ 43 \cdot 3 &= 129, \\ 161 - 129 &= 32, \\ 100 \cdot 32 + 69 &= 3269, \\ 467 \cdot 7 &= 3269, \\ 3269 - 3269 &= 0.\end{aligned}$$

Separate the number into periods: 5'61'69. There are three, so the result should have three digits. Use the same algorithm, repeating steps 4–6 until the last period has been used.

Answer: $\sqrt{56169} = 237$

Notes

1. The algorithm also works for terminating decimals. Separate the integer and fractional parts into periods independently, moving left for the integer part and right for the fractional part from the decimal comma.
2. If the root cannot be extracted exactly, put a comma after the last digit of the given number and create further periods, each consisting of 00. The extraction process is then infinite and is stopped when the required accuracy is reached.
3. A similar method can calculate square roots in any numeral system.

Example 3. Calculate the square root of 421.4809

$$\begin{aligned}\sqrt{421.4809} &= 20.53, \\ 4 - 2^2 &= 4 - 4 = 0, \\ 100 \cdot 0 + 21 &= 21, \\ 40 \cdot 0 &= 0, \\ 21 - 0 &= 21, \\ 100 \cdot 21 + 48 &= 2148, \\ 405 \cdot 5 &= 2025, \\ 2148 - 2025 &= 123, \\ 100 \cdot 123 + 9 &= 12309, \\ 4103 \cdot 3 &= 12309, \\ 12309 - 12309 &= 0.\end{aligned}$$

Answer: $\sqrt{421.4809} = 20.53$

Example 4. What is the square root of 2?

$$\begin{aligned}\sqrt{2} &= 1.4142 \dots, \\ 2 - 1^2 &= 1, \\ 100 \cdot 1 &= 100, \\ 24 \cdot 4 &= 96, \\ 100 - 96 &= 4, \\ 100 \cdot 4 &= 400, \\ 281 \cdot 1 &= 281, \\ 400 - 281 &= 119, \\ 100 \cdot 119 &= 11900, \\ 2824 \cdot 4 &= 11296, \\ 11900 - 11296 &= 604, \\ 100 \cdot 604 &= 60400, \\ 28282 \cdot 2 &= 56564, \\ 60400 - 56564 &= 3836, \\ &\vdots\end{aligned}$$

Answer: $\sqrt{2} = 1,4142\dots$

Steps of the algorithm

Suppose we need the square root of a natural number n and know that it can be extracted exactly.

1. Separate n into periods, moving right to left from the last digit and including two adjacent digits in each period. If n has an even number of digits, the first, leftmost period has two digits; if it has an odd number, the first period has one. The number of periods gives the number of digits in the result.
2. Choose the largest digit whose square does not exceed the number in the first period. This digit x is the first digit of the result.
3. Square the first digit of the result and subtract its square from the first period.
4. Append, or bring down, the second period to the right of the difference. Call the resulting number A .
5. Double the part of the result found so far, obtaining a . Choose the largest digit y for which the product of y and $(10a + y)$, the number formed by appending y to a , does not exceed A . The digit y is the second digit of the result.
6. Subtract the product of y and $(10a + y)$ from A and append the third period to the right of the difference, obtaining B . Double the part of the result found so far, obtaining b . It is easier to find it by adding $(10a + y) + y = b$. Choose the largest digit z for which the product of z and $(10b + z)$ does not exceed B . The digit z is the third digit of the result.

The next step repeats step 6. Continue until the last period has been used.

Justification of the algorithm

If the number is the square of a two-digit number, we obtain:

$$n = (10x + y)^2 = 100x^2 + 20xy + y^2 = 100x^2 + (20x + y)y$$

This chain of equalities reflects the operations described in the algorithm's steps.

If the number is the square of a three-digit number, we obtain:

$$n = (100x + 10y + z)^2 = 10000x^2 + 2000xy + 100y^2 + 200xz + 20yz + z^2 = 10000x^2 + 10000xz + 10000yz + 10000z^2 + 2000xy + 100y^2 + 200xz + 20yz + z^2$$

Analogous operations apply to a number with any number of digits.

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